

**Instructions:**

Please write your answers on separate paper. Please write clearly and legibly, using a large font and plenty of white space (I need room to put my comments). Staple all your pages together, with your problems in order, when you turn in your exam. Please don't write under the staple. Make clear what work goes with which problem. Put your name or initials on every page. To get credit, you must show adequate work to justify your answers. If unsure, show the work. Simplify your answers, except you may leave any number of more than three digits (like  $123 \times 456$  or  $15^4$ ) unsimplified. No outside materials are permitted on this exam – no notes, papers, books, calculators, phones, smartwatches, or computers – only pens and pencils, and your coursepack. You may use any result in the coursepack (whether boxed or an exercise). However, you must cite it, and you may not use it to prove itself (or a portion/special case of itself). The first nine problems are out of 10 points each (your score will be 5-10 points); the last seven problems are out of 16 points each (your score will be 8-16 points). The maximum possible score ( $9 \times 10 + 7 \times 16$ ) is 202/200 (101%). You have 120 minutes.

**Problems out of 10 points:**

1. To play a particular card game, the dealer shuffles a standard 52 card deck and creates a stack of 10 cards for the player (the player will later draw from the top of this stack). Use the twelfold way to compute how many possible 10-card stacks are possible.
2. Compute both the number of partitions of 10 into 4 parts, and the number of compositions of 10 into 4 parts.
3. Prove the formula  $\Delta(F(n)G(n)H(n)) = (\Delta F(n))G(n)H(n) + F(n+1)(\Delta(G(n))H(n) + F(n+1)G(n+1)(\Delta(H(n))))$ .
4. Let  $T_3 = \{(1\ 2), (1\ 2\ 3 \cdots n)\}$ , a set with just two permutations. Prove that  $\langle T_3 \rangle = S_n$ .
5. Prove that  $[a-1] = \binom{a}{2}$ .
6. Find, with proof, a formula for  $[a-2]$ .
7. Let  $n \geq 4$  and consider  $S_n$ . Let  $C$  be an unknown set of colors with  $|C| = m$ . Determine  $|\text{Fix}(\tau)|$  for every  $\tau \in \langle \sigma \rangle$ , where  $\sigma = (1\ 2\ 3\ 4)$ .
8. We color the edges of an equilateral triangle from an unknown set of colors  $C$  with  $|C| = m$ . Determine how many different colorings there are, up to rotation.
9. We color the edges of a square from the set of colors  $C = \{\text{red, white, blue}\}$ . Determine how many different colorings there are, up to rotation and reflection.

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**Problems out of 16 points:**

10. We are given  $n + 1$  integers from  $[2n]$ . Prove that there is at least one pair of integers where one divides the other.
11. Let  $a, b$  be integers with  $a \geq b \geq 1$ . Prove that  $\{a \atop b\} = \frac{b^a}{b!} - \sum_{i=0}^{b-1} \frac{\{a \atop i\}}{(b-i)!}$ .  
HINT: You may use exercise 1.27, if you cite it.
12. Prove the Chu-Vandermonde identity: Let  $i \in \mathbb{N}_0, a, b \in \mathbb{R}$ . Then  $\binom{a+b}{i} = \sum_{j=0}^i \binom{a}{j} \binom{b}{i-j}$ .
13. Use generating functions to solve the following recurrence:  
 $a_0 = -1, a_1 = 0, a_2 = 3, a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3} \ (n \geq 3)$ .
14. Vadim takes an alphabetized list of his  $n$  students, and splits this list into nonempty sublists (each sublist consists of consecutive names). Then, he chooses a (possibly empty) subset of these newly formed sublists to get extra credit. Find a generating function that counts the number of ways he can do this. Optional: for up to 2 points extra credit, find the closed form.
15. Compute  $\sum_{n \geq 0} \frac{n^2}{4^n}$ .
16. Consider the “square torus”. It has four identical rectangular faces on the outside, four identical square faces on the interior, and two identical faces top and bottom. Those two faces are each a  $3 \times 3$  square with the middle  $1 \times 1$  square missing. We want to color all ten faces white or black. Use Burnside’s Lemma to determine how many ways are there to do so, up to rotation (in 3d space). See below an image of the square torus, as well as one that has been colored.

